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Calculating the infrared central charges for perturbed minimal models: improving the RG perturbation

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Abstract

We illustrate a method for improving Renormalisation Group improved perturbation theory by calculating the infra-red central charge of a perturbed conformal field theory. The additional input is a dispersion relation that exploits analyticity of the energy-momentum tensor correlator.

1 Introduction

The purpose of this letter is to describe a method for improving a Renormalisation Group improved perturbative calculation. We will study a two dimensional quantum field theory off criticality. In the two scaling limits, the infrared (IR) and the ultraviolet (UV), it becomes a conformal field theory characterised by the Virasoro central charge. The field theory we look at has the minimal models $\mathcal{M}(m)$ as scaling limits. Our approximation method obtains a value for the infrared central charge c_{IR} using perturbative information around the ultraviolet limit of the theory. In [1, 2, 3] c_{IR} was calculated in perturbation theory in the limit when $m \to \infty$. This is valid because then the UV and IR fixed-points are arbitrarily close in the coupling constant space. The renormalisation group (RG) eigenvalue $y = \frac{4}{m+1}$ of the perturbation vanishes as $m \to \infty$ and the perturbation thus becomes marginal, and hence the field theory does not move away from the UV fixed-point. As well as for large values of m our approximation also applies for small values, $(m > 10)$, and we can test it against the exact result $c_{IR} = c_{\mathcal{M}(m-1)}$ when $c_{UV} = c_{\mathcal{M}(m)}$. In the limit $m \to \infty$ we obtain the perturbative result reported in [1, 2, 3]. To improve upon standard perturbative techniques requires some additional input. For this we will exploit the analyticity of the energy-momentum correlator $\langle TT\rangle$ in a complex scale parameter using it to construct a dispersion relation. In section 2 we describe this dispersion relation. Section 3 gives the results from perturbative CFT and the RG improvements, and in section 4 we illustrate our approximation with numerical results.

2 The dispersion relation

We will now construct a dispersion relation that relates the infra-red and ultra-violet behaviour of the energy-momentum tensor two-point function. The correlator of two energy-momentum tensors can be written using the Källen-Lehman spectral representation [4] as

$$\langle T_{zz}(z,\bar{z})T_{zz}(0,0)\rangle = \frac{\pi}{48}\int_0^\infty d\mu\,\tilde{c}(\mu)\int\frac{d^2p}{(2\pi)^2}\frac{e^{\frac{i}{2}(p\bar{z}+\bar{p}z)}}{p\bar{p}+\mu^2}\,\bar{p}^4 \equiv \frac{F(|z|^2)}{2\,z^4}$$

where we use the usual complex variables $z,\bar{z}$, and $\tilde{c}(\mu)d\mu$ is the spectral density of the QFT which represents the density in degrees of freedom at the mass μ. Integrating over p gives

$$F(s) = \frac{1}{48}\int_0^\infty d\mu\,\tilde{c}(\mu)\mu\sqrt{s}\left((\mu^3 s^{3/2}+24\mu\sqrt{s})K_0(\mu\sqrt{s})+(8\mu^2 s+48)K_1(\mu\sqrt{s})\right) \qquad (1)$$

where $s\in\mathbb{R}_+$ and K_0, K_1 are the modified Bessel functions. In [4] it was shown using the spectral representation that $c_{UV} = \int_0^\infty d\mu\tilde{c}(\mu)$ and $c_{IR} = \lim_{\epsilon\to 0}\int_0^\epsilon d\mu\tilde{c}(\mu)$. From the properties of the modified Bessel functions it then follows from (1) that $F(s)$ has the limits

$$F(s) \to \begin{cases} c_{UV} & \text{for } s\to 0_+, \\ c_{IR} & \text{for } s\to\infty. \end{cases} \qquad (2)$$

In [5] we prove in detail that the expression (1) provides an analytic continuation from real positive values of s to the complex plane cut along the negative real axis, as might be expected from the analyticity of the Bessel functions themselves. From now on we take s to be in the complex plane with a thin wedge about the negative real axis removed (so $-\pi+\epsilon < \arg(s) < \pi-\epsilon$). From (1) it also follows that $F(s)$ has the limits (2) for all $s\to 0$ and $|s|\to\infty$ from the cut complex plane, so $\lim_{|s|\to\infty}F(s) = c_{IR}$ and $F(0) = c_{UV}$.

The following contour integral in the cut complex plane therefore vanishes by analyticity of $F(s)$

$$\frac{1}{2\pi i}\int_C ds\,\frac{e^{\rho/s}}{s}F(s) = \frac{1}{2\pi i}\left(\int_{C_0}ds + \int_{C_1}ds + \int_{C_2}ds\right)\frac{e^{\rho/s}}{s}F(s) = 0, \qquad (3)$$

where the contour C is given in figure 1. The contribution from the large circle C_2, where $s = r_{IR}e^{i\theta}$ for $\theta\in[\pi-\epsilon,-\pi+\epsilon]$, picks out the value $(-1+\frac{\epsilon}{\pi})\lim_{|s|\to\infty}F(s) = (-1+\frac{\epsilon}{\pi})c_{IR}$ in the limit $r_{IR}\to\infty$. This is seen by writing the integral as an angular integral over θ, the limit $r_{IR}\to\infty$ can then be taken before the integral as it follows

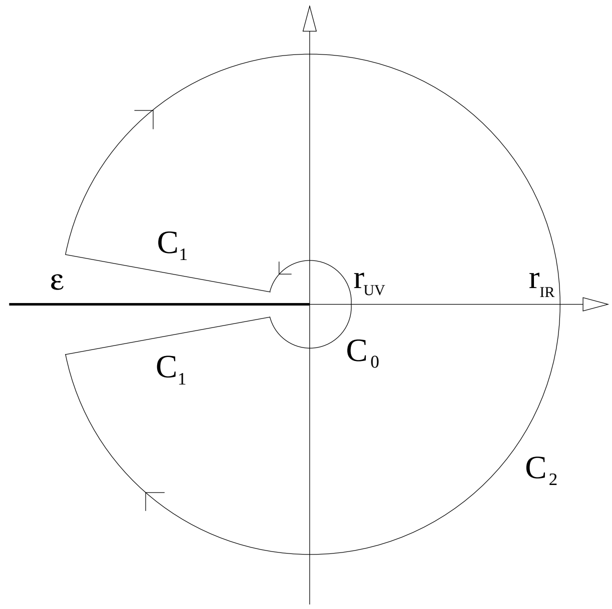

Figure 1: Integration contour C in the cut complex plane.

from (1) that the integrand is bounded by a constant in the cut plane. For ϵ small the infrared central charge can thus be calculated from an integral over the small circle C_0 and an integral over the contour C_1 which we denote $\mathrm{Cut}(\rho)$

$$c_{IR} = \frac{1}{2\pi i}\int_{C_0} ds\ \frac{e^{\rho/s}}{s} F(s) + \mathrm{Cut}(\rho) = I(\rho) + \mathrm{Cut}(\rho). \tag{4}$$

The exponential factor in the integrand suppresses the contribution of the cut for large positive values of ρ and $\lim_{\rho\to\infty}\mathrm{Cut}(\rho) = 0$ so that $c_{IR} = \lim_{\rho\to\infty} I(\rho)$. To show this we divide the $|s|$ interval into $[r_{UV}, 1]$ and $[1, r_{IR}]$ and note that $F(s)$ is finite in the unit disk hence the contribution in the lower region is bounded by the function $k\frac{e^{-\rho}}{\rho}$ for some $k \in \mathbb{R}_+$. In the other region, using the properties of the Bessel functions, the integral is seen to be bounded by the expression $k \int d\mu \tilde{c}(\mu)\mu^{7/2} \int_1^{r_{IR}} dr e^{-\rho\cos(\epsilon)/r} r^{3/4} e^{-\mu\sqrt{r}\sin\epsilon/2}$ which is uniformly convergent in ρ, the details are shown in [5].

For small s, $F(s)$ and hence $I(\rho)$ are determined by the ultra-violet behaviour. Close to the UV fixed-point we may describe this by perturbation theory with running coupling constant $\bar{g}(s)$, vanishing as $s \to 0$. We denote the perturbative approximation to the integral in (4) by $I_n(\rho)$ where $F(s)$ is approximated to n-th order by the perturbative expression $F_n(s)$. If analyticity is not spoilt by the perturbative expansion (as may be seen by inspection of the results in the next section) then simply substituting I_n for I in this limit would yield $c_{IR} \approx \lim_{\rho\to\infty} I_n(\rho) = \lim_{s\to\infty} F_n(s)$, which is just the perturbative estimate of the central charge, c^*_{IR}. Since the scale dependence of the theory can be absorbed in the scaling of the running coupling constant $\bar{g}(s)$ we can write $F_n(s) = \Phi_n(\bar{g}(s))$, so that if g^*_{IR} is the first non trivial zero of the perturbative β-function i.e. $\bar{g}(s) \to g^*_{IR}$ for $s \to \infty$, then $\lim_{\rho\to\infty} I_n(\rho) = \Phi_n(g^*_{IR})$. However, we wish to improve on this result. Both $I_n(0)$ and $I(0)$ equal c_{UV}. For small enough ρ,

$I_n(\rho)$ provides a good approximation to $I(\rho)$, since the power series expansion of $I(\rho)$ is controlled by the small s expansion of $F(s)$ for which perturbation theory is good. For larger values of ρ, $I(\rho)$ and $I_n(\rho)$ part company, tending to c_{IR} and c^*_{IR} respectively. If $c_{IR} < c^*_{IR}$ and if the region where $I_n(\rho)$ is a good approximation to $I(\rho)$ is large enough, then $I_n(0) - I_n(\rho)$ will have a maximum before approaching its limiting value of $c_{UV} - c^*_{IR}$. It is this maximum that we will use to provide a better estimate of $c_{UV} - c_{IR}$, since this occurs at the largest value of ρ for which $I_n(\rho)$ is a reasonable approximation to $I(\rho)$ and the true value of $c_{UV} - c_{IR}$ is given by $I(0) - I(\infty)$. As we will see this is the case for the minimal models considered in the next section.

3 The perturbative calculation

In the CFT $\mathcal{M}(m)$ the primary field $\phi_{(1,3)}$ is a relevant field which RG trajectory is a geodesic in the coupling constant space [3]. The QFT

$$S = S_{\mathcal{M}(m)} - \lambda_0 \int d^2x \; \phi_{(1,3)}(x). \tag{5}$$

therefore interpolates between the UV fixed-point ($\lambda_0 = 0$) given by $\mathcal{M}(m)$ and the IR fixed-point which is given by the CFT $\mathcal{M}(m-1)$. This can be seen from perturbative arguments in the $m \to \infty$ limit [1, 2, 6] and the general case is argued for in [7] using thermo-dynamical Bethe ansatz methods.

When λ_0 is small we can use perturbation theory to calculate quantities in the theory (5). Calling the bare field $\phi_{(1,3)} = \phi$ one gets [6] to order λ_0

$$\begin{aligned} \langle \phi(x)\phi(0) \rangle &= \frac{\langle \phi(x)\phi(0) e^{\lambda_0 \int d^2x' \phi(x')} \rangle_{\mathcal{M}(m)}}{\langle e^{\lambda_0 \int d^2x' \phi(x')} \rangle_{\mathcal{M}(m)}} \\ &= \frac{1}{|x|^{2(2-y)}} \left(1 + \lambda_0 \frac{4\pi b(y) A(y)}{y} |x|^y + O(\lambda_0^2) \right), \end{aligned}$$

where $A(y) = \frac{\Gamma(1-y)\Gamma(1+y/2)^2}{\Gamma(1-y/2)^2\Gamma(1+y)} = 1 + O(y^3)$ and $b(y)$ is the operator product expansion coefficient $C^{\phi}_{\phi\phi}$ which can be calculated in the Coulomb gas representation of minimal models using the formulas in [8]. $b(y)$ is given by

$$b(y)^2 = \frac{16}{3} \frac{(1-y)^4}{(1-y/2)^2(1-3y/4)^2} \left(\frac{\Gamma(1+y/2)}{\Gamma(1-y/2)} \right)^4 \left(\frac{\Gamma(1-y/4)}{\Gamma(1+y/4)} \right)^3 \left(\frac{\Gamma(1-y)}{\Gamma(1+y)} \right)^2 \left(\frac{\Gamma(1+3y/4)}{\Gamma(1-3y/4)} \right) = \frac{16}{3} + O(y).$$

$y = \frac{4}{m+1}$ is the RG eigenvalue for the perturbation $\phi_{(1,3)}$. With the renormalisation conditions $\langle \phi(x,g)\phi(0,g) \rangle |_{|x|=\mu^{-1}} \equiv \mu^4$, the renormalised correlator and β-function, $\beta(g) \equiv \mu \frac{dg(\mu)}{d\mu}$, becomes to the same order [6]

$$\begin{aligned} \langle \phi(x,g)\phi(0,g) \rangle &= \frac{\mu^4}{|\mu x|^{2(2-y)}} \left(1 + \frac{4\pi A(y) b(y) g}{y} (|\mu x|^y - 1) \right), \\ \beta(g) &= -yg - \pi b(y) g^2 A(y) + O(g^3). \end{aligned}$$

Here g and $\phi(x,g)$ are the renormalised coupling and field. The RG fixed-points, the zeros of the $\beta-$function, are thus $g_{UV}=0$, $g^*_{IR}=\frac{-y}{\pi A(y)b(y)}$ and therefore $g\in(\frac{-y}{\pi A(y)b(y)},0)$ as the theory (5) lies between the two scaling limits. The trace of the energy-momentum operator Θ is the infinitesimal generator for scale transformations and it is given by [1] $\Theta(x)=2\pi\beta\phi(x)$. Then it follows from the Ward identities that

$$\begin{aligned}\partial_{\bar{z}_1}\partial_{\bar{z}_2}\langle T(z_1,\bar{z}_1)T(z_2,\bar{z}_2)\rangle &= \frac{1}{4^2}\partial_{z_1}\partial_{z_2}\langle\Theta(z_1,\bar{z}_1)\Theta(z_2,\bar{z}_2)\rangle \\ &= \frac{\pi^2}{4}\beta(g)^2\partial_{z_1}\partial_{z_2}\langle\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\rangle.\end{aligned} \tag{6}$$

Writing $\tilde{F}(\tilde{R})=2z^4\langle T(z,\bar{z})T(0,0)\rangle$ in terms of the dimensionless variable $\tilde{R}=\mu^2 z\bar{z}$, then (6) becomes

$$\frac{\partial^2}{\partial\tilde{R}^2}\tilde{F}(\tilde{R})=\frac{\pi^2\beta^2}{4\mu^4}\tilde{R}^2\frac{\partial^2}{\partial\tilde{R}^2}\langle\phi(\tilde{R})\phi(0)\rangle. \tag{7}$$

The limiting values of $\tilde{F}(\tilde{R})$ are given by (2), using these the solution to (7) becomes

$$\begin{aligned}\tilde{F}(\tilde{R}) = c_{UV} + \frac{\pi^2 g^2\tilde{R}^y}{2}\Bigg(\frac{y(2-y)(3-y)}{y-1} + 2\pi A(y)b(y)g\Bigg(\frac{(2-y)(3-y)}{1-y} \\ +\tilde{R}^{\frac{y}{2}}\frac{(3y-4)(3y-6)}{3(\frac{3}{2}y-1)}\Bigg)\Bigg).\end{aligned}$$

The function $F(s)$ is determined as $F(s)=\tilde{F}(\tilde{R})|_{\tilde{R}=1,g=\bar{g}(s)}$. The theory is thus fixed at the renormalisation point and all scale dependence is moved into the running coupling constant [6]

$$\bar{g}(s)=\frac{gs^{\frac{y}{2}}}{1-\frac{\pi A(y)b(y)g}{y}(s^{\frac{y}{2}}-1)}. \tag{8}$$

We thereby get our 1 loop RG improved approximation of $F(s)$

$$\begin{aligned}F_1(s) = c_{UV} + \frac{\pi^2}{2}\bar{g}^2(s)\Bigg(\frac{y(2-y)(3-y)}{y-1} \\ +2\pi A(y)b(y)\bar{g}(s)\Bigg(\frac{(2-y)(3-y)}{1-y}+\frac{(3y-4)(3y-6)}{3(\frac{3}{2}y-1)}\Bigg)\Bigg).\end{aligned} \tag{9}$$

To apply our approximation based on the dispersion relation of the previous section to calculate $\Delta c=c_{UV}-c_{IR}$ we now have to find the maximum of

$$I_1(0)-I_1(\rho)=c_{UV}-\frac{1}{2\pi i}\int_{C_0}ds\frac{e^{\rho/s}}{s}F_1(s). \tag{10}$$

4 Results

We first note that $F_1(s)$ has the correct value in the perturbative limit $y \to 0$. From (9) it follows that $\Delta c_{pert} = c_{UV} - c^*_{IR} = \lim_{s\to 0}(c_{UV} - F_1(s))$ is given by

$$\begin{aligned}
\Delta c_{pert} = & -\frac{\pi^2}{2}(g^*_{IR})^2 \left(\frac{y(2-y)(3-y)}{y-1} \right. \\
& \left. +2\pi A(y)b(y)g^*_{IR} \left(\frac{(2-y)(3-y)}{1-y} + \frac{(3y-4)(3y-6)}{3(\frac{3}{2}y-1)} \right) \right) \\
= & \frac{3y^3}{16} + O(y^4),
\end{aligned} \tag{11}$$

and the exact value is

$$\Delta c_{exact} = c(m) - c(m-1) = \frac{12}{m(m^2-1)} = \frac{3y^3}{2(2-y)(4-y)} = \frac{3y^3}{16} + O(y^4). \tag{12}$$

To calculate Δc for finite m we have to calculate $I_1(\rho)$. The running coupling (8) can be rewritten as

$$\bar{g}(s) = \frac{gs^{\frac{y}{2}}}{1 - \frac{\pi A(y)b(y)g}{y}(s^{\frac{y}{2}} - 1)} = \frac{g^*_{IR}|\tilde{g}|s^{y/2}}{1 + |\tilde{g}|s^{y/2}}, \qquad \tilde{g} = \frac{g}{g - g^*_{IR}} \in (-\infty, 0). \tag{13}$$

A variable change $s' = s|\tilde{g}|^{2/y}$ in $I_1(\rho)$ then leads to

$$I_1(\rho, g) = \frac{1}{2\pi i}\int_{C_0} ds \frac{e^{\rho/s}}{s} F_1(s) = \frac{1}{2\pi i}\int_{C'_0} ds' \frac{e^{\rho'/s'}}{s'} F_1(s') = \tilde{I}_1(\rho') \tag{14}$$

where now all dependence of the renormalised coupling g is moved into $\rho' = \rho|\tilde{g}|^{2/y}$ and the radius of C'_0: $r'_{UV} = r_{UV}|\tilde{g}|^{2/y}$ as now $\bar{g}(s') = \frac{g_{IR^*}}{1+s'^{y/2}}$. The integral $\tilde{I}_1(0) - \tilde{I}_1(\rho')$ can be calculated numerically and in figure 2 we have plotted it against $\log \rho'$ for the case of $m = 14$. For large ρ' it tends to the perturbative value 0.0025, but has a maximum value of 0.00367 which provides a better approximation to the true value of 0.00440. The dashed line indicates how we expect $\tilde{I}(0) - \tilde{I}(\rho')$ to behave. We have approximated Δc for different values of m by the maximum of $\tilde{I}_1(0) - \tilde{I}_1(\rho')$ and we denote this approximation as Δc_{approx}. The numbers Δc_{approx} have been obtained by a numerical integration using a NAG mark 18 Fortran Library quadrature routine. In figure 3 the error in Δc_{approx} and Δc_{pert} compared with Δc_{exact} is plotted against m. The errors are scaled with $m(m^2-1)$ so that all points are distinguishable on the same plot. The figure shows that the error in the approximation Δc_{approx} is more than a factor 2 smaller than the error in the perturbative value Δc_{pert} in the region plotted.

For values of m smaller than 11 perturbation theory begins to breakdown indicated by the RG improved result turning negative (violating unitarity). Our approximation whilst still positive for m close to 10 also becomes poorer since it is based on the RG expression. Also, note that both Δc_{approx} and Δc_{pert} approaches Δc_{exact} faster than the asymptotic value $3y^3/16$ in the limit $m \to \infty$.

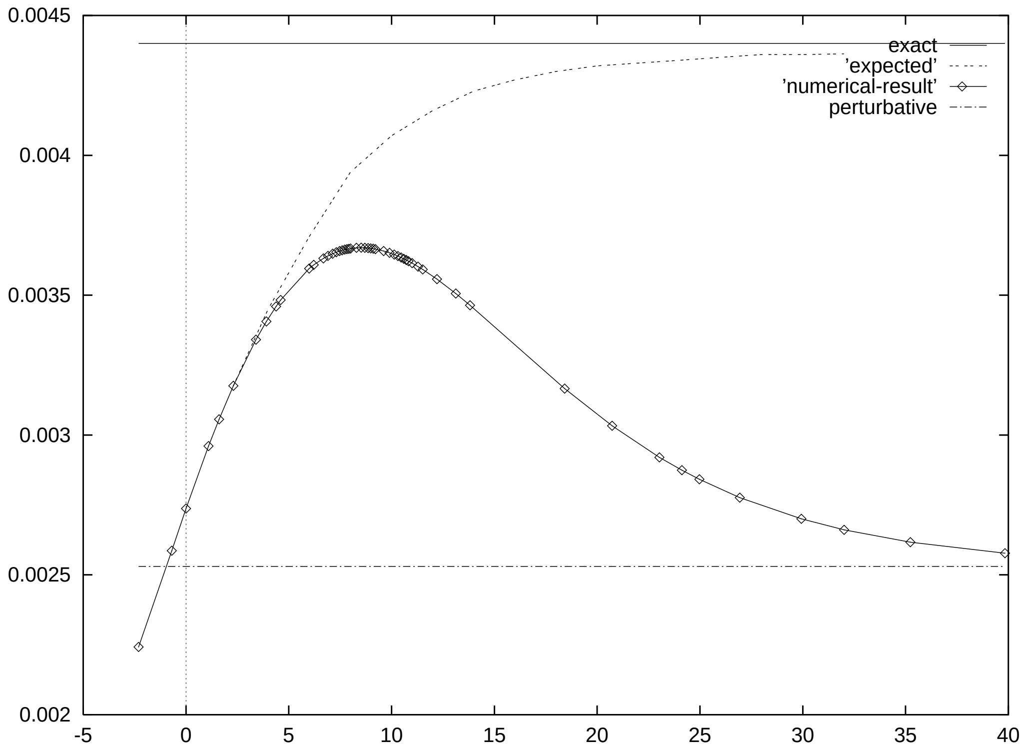

Figure 2: The numerical-result $\tilde{I}_1(0) - \tilde{I}_1(\rho')$ against $\log \rho'$, also plotted is the exact value Δc_{exact} and the RG improved perturbative result Δc_{pert} all for $m = 14$. The dashed line is the expected behaviour of $\tilde{I}(0) - \tilde{I}(\rho')$.

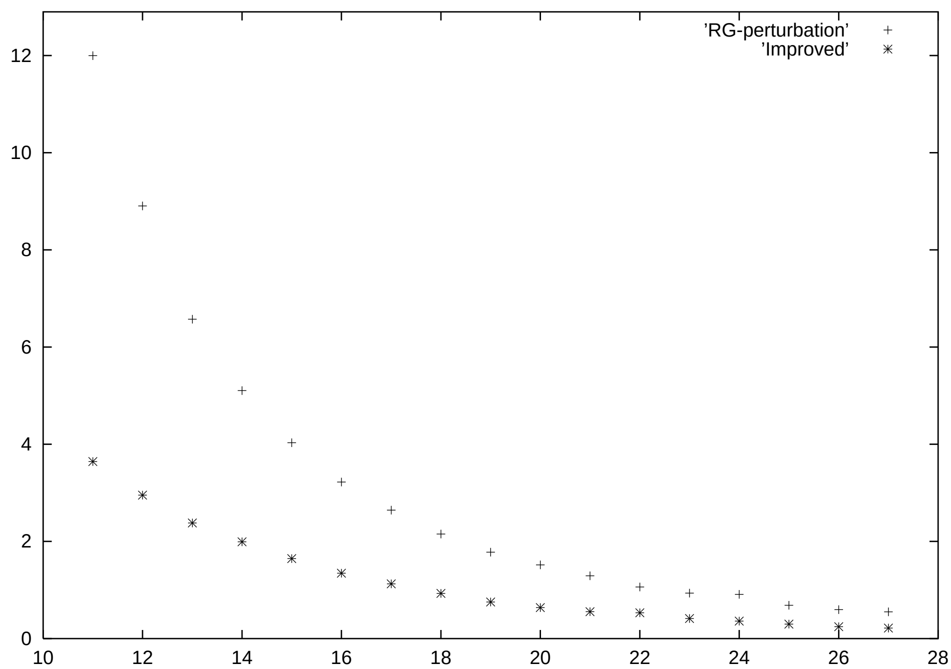

Figure 3: $(\Delta c_{exact} - \Delta c_{approx})m(m^2-1)$ and $(\Delta c_{exact} - \Delta c_{pert})m(m^2-1)$ against m.

5 Conclusion

We have described an approximation method to obtain the IR central charge of the minimal models valid for $m > 10$. The method improves upon the RG improved calculation by exploiting the analyticity of the correlator of the energy-momentum tensor. This analyticity is a property both of the exact correlator and the RG improved perturbative estimate of it, [5], which we have used in our calculation.

Our approximation is correct in the perturbative limit $m \to \infty$. For smaller m we obtained the values shown in figure 3 together with the RG improved perturbative result. This figure demonstrates that the approximation is significantly better than the RG improved perturbative result, e.g. for $m = 18$ it has a 7.8% relative deviation from the exact result whereas the RG perturbative result deviates by 18.0%. The analyticity in the complex scale parameter s of the energy-momentum tensor two-point function is in fact an ubiquitous property of correlation functions in quantum field theory, having its origin in the hermiticity of the Hamiltonian. Consequently we expect our approach to be applicable beyond the specific calculation we have used to illustrate it here, to other RG improved perturbative calculations of Green's functions.

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